Constraints on perturbative and non perturbative completions

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Preulde

Our world up to now looks perturbative (S-matrix exists)

What can we expect in the UV?

- Continues to be perturbative, with IR degrees of freedom still present in the UV (Four fermi $\rightarrow$ Electro Weak) S-matrix exists
- Becomes non-perturbative, with IR degrees of freedom still present in the UV (Quantum Gravity) S-matrix may exists
- Becomes non-perturbative, with IR degrees of freedom emerging as bound state (Pions $\rightarrow$ QCD) S-matrix exists
- Becomes a CFT S-matrix does not exists, even non-lagrangian

Preulde

Our world up to now looks perturbative (S-matrix exists)

What can we expect in the UV?

- Continues to be perturbative, with IR degrees of freedom still present in the UV

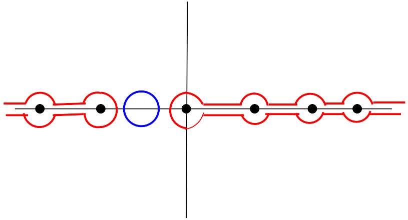
  Low energy  interactions, perturbative completion is highly constrained

- Becomes a CFT:

  Vacuum manifold $\rightarrow$ spontaneous symmetry breaking $\rightarrow$ Goldstone bosons (EFT)
  S-matrix does exists

What is perturbative completion?

- The UV degrees of freedom appears while the theory is still weakly coupled
  - The S-matrix only have poles, no branch cuts



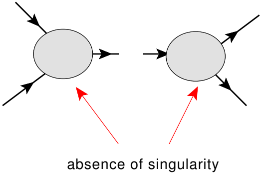
  - The new degrees of freedom (glue balls and mesons in large N YM)
- The high energy fixed angle scattering is improved

General solution:

$$M(s,t) = \frac{n(s,t)}{stu} \rightarrow \frac{1}{stu} f(s,t,m_i) \equiv \frac{1}{stu} \frac{n'(s,t,m_i)}{\prod_{i=1}^{\infty}(s-m_i^2)(t-m_i^2)(u-m_i^2)}$$

But locality requires

$$f(s,t,m_i)\Big|_{s=m_i^2} \rightarrow$$



All double poles must have no residue

$$\begin{aligned} s=a,\ t=b &\rightarrow & n'(a,b)=0 \\ s=a,\ u=b &\rightarrow & n'(a,-a-b)=0 \\ s=a,\ t=b &\rightarrow & n'(-a-b,b)=0 \end{aligned}$$

$n'(s,t,m_i)$ is over constrained bounded polynomial

Any system with low energy  interactions can only be completed with an infinite tower

What can we expect?

- As $s \to \infty$, for $t < 0$ causality requires Camanho, Edelstein, Maldacena, Zhiboedov

$$M(s,t)|_{s\to\infty} \sim s^{2+\alpha(t)}, \alpha(t) < 0$$

- For $s, t >> 0$ the amplitude behaves as Caron-Huot, Komargodski, Sever, Zhiboedov

$$M(s,t)|_{s\to\infty} \sim s^{j(t)}, \quad j(t) \sim t$$

- For gravity, at low energies we have

$$M(h_1^-, h_2^-, h_3^+, h_4^+) = \frac{\langle 12\rangle^4 [34]^4}{stu}$$

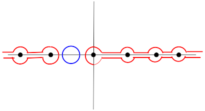
We expect

$$M(s,t) = \langle 12\rangle^4 [34]^4 f(s,t,m_i), \quad f(s,t,m_i)|_{s\to\infty} \sim s^a$$

with $a < -2$

- Then, for fixed t^*

$$f(s,t^*) = \int \frac{dv}{v-s} f(v,t^*)$$



$$= \left(\sum_i \frac{r[t]_{s=m_i^2}\,(t+2m_i^2)}{(s-m_i^2)(s+t+m_i^2)} + \frac{r[t]_{s=0}}{s(-s-t)t} \right)$$

as $s >> 0$ this is just a polynomial in t, yet must contain poles in t infinite higher spin !

General solution:

$$M(s,t) = \langle 12\rangle^4 [34]^4 f(s,t,m_i) = \langle 12\rangle^4 [34]^4 \frac{n(s,t)}{\prod_{i=1}^{\infty}(s-m_i^2)(t-m_i^2)(u-m_i^2)}$$

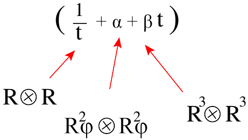
Let

$$n(s,t) \sim \frac{\prod_{\{i,j\}}(s+m_i^2+m_j^2)(t+m_i^2+m_j^2)(u+m_i^2+m_j^2)}{stu\prod_i(s-m_i^2)(t-m_i^2)(u-m_i^2)}$$

We're done, this is string theory!

Massless residues controlled by the interaction of three massless particles $\leftarrow$ highly constrained!

One only has R, $R^2\phi$, R^3. This implies that the massless residue, for $s = 0$, must be



On the other hand the massless residue of our ansatz is

$$M(s,t)|_{s=0} \sim \langle 12\rangle^4[34]^4 \frac{\prod_{\{i,j\}}(m_i^2+m_j^2)(t+m_i^2+m_j^2)(-t+m_i^2+m_j^2)}{\prod_{i=1}^{\infty}(m_i^2)(t-m_i^2)(t+m_i^2)}$$

We must have for any two pair of $\{i,j\}$ there exists an m_k^2 such that $m_i^2+m_j^2=m_k^2$

$$m_i^2 = \frac{1}{\alpha'}\{1,2,3,\cdots\}$$

$$m_i^2 = \frac{1}{\alpha'}\{1, 2, 3, \cdots\}$$

We thus find a simple solution:

$$M(s,t) = \frac{\langle 12\rangle^4[34]^4}{stu}\prod_{i=1}^{\infty}\frac{(s+i)(t+i)(u+i)}{(s-i)(t-i)(u-i)} = \langle 12\rangle^4[34]^4\frac{\Gamma[1-t]\Gamma[1-s]\Gamma[1-u]}{\Gamma[1+t]\Gamma[1+s]\Gamma[1+u]}$$

This is nothing but the closed superstring amplitude!

In fact this is the universal piece in all perturbative string completion:

$$\begin{aligned}
\text{Super} \quad f(s,t) &= \frac{\Gamma[1-s]\Gamma[1-u]\Gamma[1-t]}{\Gamma[1+s]\Gamma[1+u]\Gamma[1+t]}\left(\frac{-1}{stu}\right) \\
\text{Heterotic} \quad f(s,t) &= \frac{\Gamma[1-s]\Gamma[1-u]\Gamma[1-t]}{\Gamma[1+s]\Gamma[1+u]\Gamma[1+t]}\left(\frac{-1}{stu}+\frac{1}{s(1+s)}\right) \\
\text{Bosonic} \quad f(s,t) &= \frac{\Gamma[1-s]\Gamma[1-u]\Gamma[1-t]}{\Gamma[1+s]\Gamma[1+u]\Gamma[1+t]}\left(\frac{-1}{stu}+\frac{2}{s(1+s)}-\frac{tu}{s(1+s)^2}\right)
\end{aligned}$$

Each additional term corresponds to the presence of $R^2\phi$, R^3.

Does this mean perturbative string is the only solution?

Not yet

Consider the following deformation:

$$\frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+s]\Gamma[1+t]\Gamma[1+u]}\left(1+\epsilon\frac{stu}{(s+1)(t+1)(u+1)}\right)$$

Consistent for $0<\epsilon<1$!

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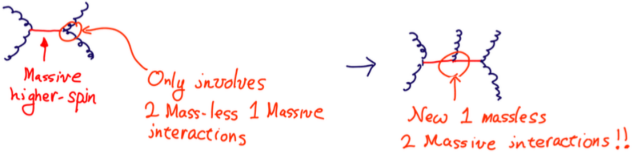
Consistent for $0<\epsilon<1$!

We've seen this before:

- In the early days there was the Lovelace-Shapiro model with intercept $\alpha_0 = \frac{1}{2}$ that gave a consistent four-point amplitude, but no *n*-pt generalisation was found
- From a particle theorist point of view, this is bizarre!
- The four-point amplitude tells us the rules for  and , what could be wrong?

What might have gone wrong?



All massive interactions must be consistent as well. More precisely the three-point interactions must be such that there exists four-point local functions that can factorize into these three-points. (Exp: This is how massless higher spins are ruled out)

More over four-pt is sufficient to answer this!

What we need :

A useful way of invariantly parameterize all possible massive higher-spin interactions. → 4-D Massive Spinor Helicity

Lorentz group $SU(2) \times SU(2)$ ← Identify with $SU(2)$ little group

What might go wrong : At high E limit , the interactions are effectively massless. The only thing that are allowed to survive must be the consistent massless interactions which are already beaten to death !

Massive Interactions

Arkani-Hamed, Huang, Y-T

① States : $\epsilon^{(\mu\nu\cdots\rho)} \to k_\mu \epsilon^{\mu\nu\cdots\rho} = \epsilon_\mu{}^{\mu\nu\cdots\rho} = 0$ $\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

on-shell states are represented as SU(2) Irreps

consider $\epsilon^\mu = \epsilon^{\alpha\dot\alpha} \to \epsilon^{(\alpha\beta)} = \epsilon^{\alpha\dot\alpha}\frac{p_{\dot\alpha}{}^{\beta}}{m}$ $\left(\epsilon^{[\alpha\dot\alpha}\frac{p_{\dot\alpha}{}^{\beta]}}{m} = \epsilon^{\gamma\dot\alpha}\frac{p_{\dot\alpha\gamma}}{m}\epsilon^{\alpha\beta} = 0 \right)$

All states can be represented as symmetric tensors of SU(2) $\left(k^{\alpha\dot\alpha} = \lambda^\alpha\tilde\lambda^{\dot\alpha} + \eta^\alpha\tilde\eta^{\dot\alpha} \right)$

$$\epsilon^{(+)\alpha\beta} = \eta^\alpha\eta^\beta \qquad \epsilon^{(0)\alpha\beta} = \lambda^\alpha\eta^\beta + \lambda^\beta\eta^\alpha \qquad \epsilon^{(-)} = \lambda^\alpha\lambda^\beta$$

② Interactions : $V^{h_1h_2}_{(\alpha_1\cdots\alpha_{2L})}$ $\sim \overbrace{\lambda_1\cdots\lambda_1}^{(L+h_1-h_2)}\overbrace{\lambda_2\cdots\lambda_2}^{(L+h_2-h_1)}[12]^{L+h_1+h_2}$

2-massless 1 massive

$V^{h_1}_{(\alpha_1\cdots\alpha_{2L})(\beta_1\cdots\beta_{2L})}$

1-massless 2-massive

- Equal mass $(\lambda_{1\alpha}, \epsilon_{\alpha\beta}, x)$, $p_{2\alpha\dot\alpha}\tilde\lambda_1^{\dot\alpha} = \frac{m}{x}\lambda_{1\alpha}$
- Unequal mass $(\lambda_{1\alpha}, u_\alpha \equiv p_{2\alpha\dot\alpha}\tilde\lambda_1^{\dot\alpha})$

$V_{(\alpha_1\cdots\alpha_{2L})(\beta_1\cdots\beta_{2L})(\gamma_1\cdots\gamma_{2L})}$ $\quad p_1^{(\alpha\dot\beta}p_2^{\beta)}{}_{\dot\beta} \quad \epsilon^{\alpha\beta}$

3-massive

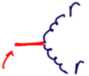
Exp:

$$V^{h_1 h_2}_{(\alpha_1 \cdots \alpha_{2L})} \sim \overbrace{\lambda_1 \cdots \lambda_1}^{(L+h_1-h_2)} \; \overbrace{\lambda_2 \cdots \lambda_2}^{(L+h_2-h_1)} \; [12]^{L+h_1+h_2}$$

2-massless 1 massive

$L = 1 \quad h_1 = +1 \quad h_2 = +1$

$$V^{11}_{\alpha_1 \alpha_2} \sim \lambda_{1\alpha_1} \lambda_{2\alpha_2} [12]^3$$



Massive - spin-1

anti-symmetric $1 \Leftrightarrow 2$

A massive vector cannot couple to two identical vectors (Yangs Theorem)

2 gravitons can consistently couple to massive spin-2

What spins can be charged? (A one massless two massive exp)

$\sim (\lambda_{1\alpha}\lambda_{1\beta} + a m x \epsilon_{\alpha\beta})$

we can consider high energy limit

renormalizable interaction only allows

for $(-\frac{1}{2} - +\frac{1}{2})$ but not $(-\frac{1}{2} - -\frac{1}{2})$

fixed $a = -1$

$$\sim \left\{ a \frac{\lambda_{1\alpha}\lambda_{1\beta}\lambda_{1\gamma}\lambda_{1\delta}}{m x} + b \begin{bmatrix} \lambda_{1\alpha}\lambda_{1\gamma}\,\epsilon_{\beta\delta} \\ \lambda_{1\alpha}\lambda_{1\delta}\,\epsilon_{\beta\gamma} \end{bmatrix} + c\ \epsilon_{\alpha\gamma}\epsilon_{\beta\delta} \right\}$$

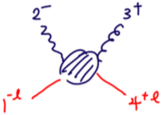
$a = 1 \quad b = -\frac{1}{2} \quad c = \frac{1}{2}$

Recall that $(- - -)$ corresponds to F^3 and $(- - 0)$ corresponds to $F^2\phi$

Lesson: massive interactions are constrained by consistency of mass-less interactions

✱ As one move to massive spin ℓ we will have $\lambda^{2\ell}$ term! ↓ (something goes wrong)

Minimal coupling leads to the following 4-pt amp as $E \to \infty$



$L = 0 \qquad A_4 = \dfrac{\langle 2|P_4 - P_1|3]^2}{(s-m^2)(u-m^2)}$

$L = \frac{1}{2} \qquad A_4 = \dfrac{\langle 2|P_4 - P_1|3]\, \langle 21\rangle [43]}{(s-m^2)(u-m^2)}$

$L = 1 \qquad A_4 = \dfrac{\langle 21\rangle^2 [43]^2}{(s-m^2)(u-m^2)}$

$L = \frac{3}{2} \qquad A_4 = \dfrac{1}{(s-m^2)(u-m^2)} \langle 2|P_4 - P_1|3]^2 \, \dfrac{\langle 42\rangle^3 [23]^3}{m^2}$

Obstruction to taking $m \to 0$

Charged $s > 1$ states cannot be fundamental particles

→ (a) bound state (b) degeneracy to the spectrum. (To be continued)

For non-perturbative completion, what can one possibly say?

In a EFT we have an infinite set of irrelevant operators

$$\mathcal{L}_{EFT} = \mathcal{L}_{marginal} + \sum_i c_i \mathcal{O}_i(\partial, \phi)$$

In general $c_i \to c_i(g, N)$

- For non-lagrangian theories c_i is simply a number!
- For theories with S-duality, $c_i(g, N)$ is constrained
- With SUSY some c_i are determined exactly

How much constraint can we impose in the IR on $\mathcal{L}_{EFT}$?

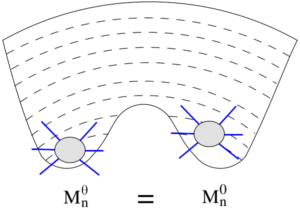
If the low lying degrees of freedom are GB $\rightarrow$ non-linearly symmetry

- How do we use the non-linear symmetry to constrain the EFT?
- Is there a systematic way to proceed with arbitrary symmetry breaking (internal and spacetime)?
- Are non-linear symmetries protected against quantum corrections?

Soft theorems

The D.O.F. for $\mathcal{L}_{EFT}$ are Goldstone bosons $\rightarrow$ Adler's zero

$$M_n(\pi_1\cdots)|_{p_1\rightarrow 0} = 0$$



$$e^{\theta G}|0\rangle = |\theta\rangle, \quad M_n^0 \equiv \langle 0|\cdots|0\rangle$$

$$M_n^\theta \equiv \langle\theta|\cdots|\theta\rangle = M_n^0 + M_{n+\pi}^0 + M_{n+\pi+\pi}^0 + \cdots$$

The U(1) goldstone bosons are derivatively coupled: $\mathcal{L}(\partial\phi)$ (Non-abelian extension see I. Low 14)

Soft theorems

Space-time symmetry breaking are different

- The generators have non-trivial commutator with P

$$[P, K] \sim D$$

  The Goldstone modes of the broken generators are derivatively related One dilaton
- For sCFT, there will be associated broken internal symmetries pions

There are multiple Goldstone modes for spontaneous space-time symmetry breaking

What does this imply for the effective action?

Soft theorems

Ward identity

$$\partial_\mu \langle J^\mu(x)\phi(x_1)\cdots\phi(x_n)\rangle = -\sum_i \delta(x-x_i)\langle\phi(x_1)\cdots\delta\phi(x_i)\cdots\phi(x_n)\rangle$$

Spontenous symmetry breaking implies $J^\mu|0\rangle = p^\mu|phys\rangle$

- LHS: performing LSZ reduction on $i = 1,\cdots,n \rightarrow \; M_n(\pi_1\cdots)|_{p_1\rightarrow 0} = 0$
- RHS: $\begin{cases} = 0 \text{ if } \delta\phi \neq |phys\rangle \\ \neq 0 \text{ if } \delta\phi = |phys\rangle \end{cases}$

Conventional spontaneous symmetry breaking: $\delta\phi = constant$ hence Adler's zero

Soft theorems

Spontaneous broken dilation and conformal boost generator leads to single dilaton,

$$[K, D] \sim K$$

The dilaton transforms linearly under the broken generator $\rightarrow$ non-vanishing soft-limits:
Boels, Wormsbecher, Y-t Wen, Di Vecchia, Marotta, Mojaza, Nohle

$$M_n\Big|_{p_n \to 0} = \left(\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(1)}\right) M_{n-1} + \mathcal{O}(p_n^2)\,,$$

$(\mathcal{S}_n^{(0)}, \mathcal{S}_n^{(1)})$ are universal soft functions

$$\mathcal{S}_n^{(0)} = \sum_{i=1}^{n-1}\left(p_i \cdot \frac{\partial}{\partial p_i} + \frac{d-2}{2}\right) - d\,,$$

$$\mathcal{S}_n^{(1)} = p_n^{\mu} \sum_{i=1}^{n-1}\left[p_i^{\nu} \frac{\partial^2}{\partial p_i^{\nu} \partial p_i^{\mu}} - \frac{p_{i\mu}}{2} \frac{\partial^2}{\partial p_{i\nu} \partial p_i^{\nu}} + \frac{d-2}{2} \frac{\partial}{\partial p_i^{\mu}}\right].$$

Soft theorems

There's more! In general CFTs with scalar moduli space has "flavor" symmetry, which will be spontaneously broken along with conformal symmetry $\rightarrow$ pions

Exp: $\mathcal{N} = 4$ SYM on Coulomb branch, 6 massless scalars (1 dilaton φ, 5 SO(6)$\rightarrow$SO(5) GBs ϕ^I)

$$A_n(\phi_1, \cdots, \phi_n^I)\big|_{p_n \rightarrow 0} = \sum_i A_{n-1}(\cdots, \delta^I O, \cdots) + \mathcal{O}(p_n^1)\,.$$

where $\delta^I \varphi = \phi^I$ and $\delta^I \phi^J = -\delta^{IJ}\varphi$.

Soft theorems

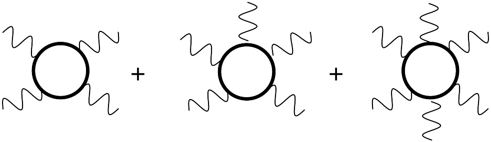
The soft theorems should be respected

- In the UV where massive D.o.F are present
- In the IR where massive D.o.F integrated away perturbatively
- In the IR where massive D.o.F integrated away non-perturbatively

Let’s check

Perturbative Verifications

The one-loop effective action of $\mathcal{N}=4$ SYM on the Coulomb branch, up to six fields



Derived from the integrand of SYM in D-dimensions (scalars: $\epsilon \cdot k_i = 0$, $\epsilon \cdot \ell = m$ for φ, $\epsilon \cdot \ell = 0$ for ϕ^I)

$$\mathcal{L}^{\mathrm{SU}(4)\ \mathrm{singlet}}_{1-\mathrm{loop}} = \frac{g^4 N}{32 m^4 \pi^2}\left(\mathcal{O}_{F^4} + \frac{\mathcal{O}_{D^4F^4}}{2^3 \times 15 m^4} - \frac{2\mathcal{O}_{D^2F^6}}{15 m^6} + \frac{\mathcal{O}_{D^4F^6}}{2^3 \times 21 m^8} - \frac{\mathcal{O}_{D^6F^6}}{2 \times 15^2 m^{10}} + \cdots\right)$$

$$\mathcal{L}^{\mathrm{Sp}(4)}_{1-\mathrm{loop}} = \frac{\partial^4\varphi^4}{16 m^4} + \frac{\partial^8\varphi^4}{960 m^8} + \frac{\partial^4\varphi^5}{4 m^6} + \frac{\partial^8\varphi^5}{480 m^{10}} - \frac{5\partial^4\varphi^6}{4 m^6} - \frac{\partial^8\varphi^6}{480 m^{10}} + \frac{\partial^{10}\varphi^6}{2^{10}3^5 m^{12}} + \frac{\partial^{12}\varphi^6}{2^{11}3^2 m^{14}} + \frac{\partial^4\varphi^2\phi'^2}{8 m^4} - \frac{5\partial^4\varphi^2\phi'^4}{4 m^6} + \frac{\partial^4\varphi^4\phi'^2}{4 m^6} + \ldots$$

Perturbative Verifications

$$\partial^4\varphi^m: \sum_{i<j} s_{ij}^2, \quad \partial^8\varphi^4: (\sum_{i<j} s_{ij}^2)^2, \quad \partial^8\varphi^5: (\sum_{i<j} s_{ij}^2)^2,$$

$$\partial^8\varphi^6: -\frac{b_1^{(4)}}{6} + \frac{5\,b_2^{(4)}}{768} + \frac{b_3^{(4)}}{36} - \frac{3\,b_4^{(4)}}{2},$$

$$\partial^{10}\varphi^6: -\frac{48\,b_1^{(5)}}{7} + \frac{36}{35}b_2^{(5)} + \frac{108}{7}b_3^{(5)} + \frac{114}{35}b_4^{(5)} + \frac{60}{7}b_5^{(5)},$$

$$\partial^{12}\varphi^6: \frac{433}{1350}b_1^{(6)} - \frac{58}{2025}b_2^{(6)} + \frac{20}{9}b_3^{(6)} + \frac{117}{35}b_4^{(6)} - \frac{184}{945}b_5^{(6)},$$

$$-\frac{74}{45}b_6^{(6)} + \frac{334}{315}b_7^{(6)} + \frac{177}{35}b_8^{(6)} - \frac{64}{63}b_9^{(6)} + \frac{104}{105}b_{10}^{(6)}$$

$$\partial^4\varphi^2\phi^2: s_{12}^2 - s_{13}^2 - s_{23}^2, \quad \partial^4\varphi^2\phi^4: b^{(2)}_{1,S_2\times S_4} - b^{(2)}_{2,S_2\times S_4} + b^{(2)}_{3,S_2\times S_4} - \frac{8}{5}b^{(2)}_{4,S_2\times S_4}$$

$$\partial^4\varphi^4\phi^2: b^{(2)}_{1,S_2\times S_4} - b^{(2)}_{2,S_2\times S_4} + b^{(2)}_{3,S_2\times S_4} - 8b^{(2)}_{4,S_2\times S_4}$$

$$\begin{aligned}
b_1^{(4)} &= s_{12}^4 + \mathcal{P}_6, \quad b_2^{(4)} = (s_{12}^2 + \mathcal{P}_6)^2, \quad b_3^{(4)} = s_{12}^2 s_{13}^2 + \mathcal{P}_6, \\
b_4^{(4)} &= s_{123}^4 + \mathcal{P}_6, \quad b_1^{(5)} = s_{12}^5 + \mathcal{P}_6, \quad b_2^{(5)} = s_{12}^2 s_{123}^3 + \mathcal{P}_6, \\
b_3^{(5)} &= s_{12}^2 s_{13}^3 + \mathcal{P}_6, \quad b_4^{(5)} = s_{12}^2 s_{34}^2 + \mathcal{P}_6, \quad b_5^{(5)} = s_{123}^5 + \mathcal{P}_6 \\
b_1^{(6)} &= s_{12}^6 + \mathcal{P}_6, \quad b_2^{(6)} = s_{123}^6 + \mathcal{P}_6, \quad b_3^{(6)} = s_{12}^4 s_{13}^2 + \mathcal{P}_6, \\
b_4^{(6)} &= s_{12}^4 s_{34}^2 + \mathcal{P}_6, \quad b_5^{(6)} = s_{12}^3 s_{13}^3 + \mathcal{P}_6, \quad b_6^{(6)} = s_{12}^3 s_{34}^3 + \mathcal{P}_6, \\
b_7^{(6)} &= s_{12}^2 s_{123}^4 + \mathcal{P}_6, \quad b_8^{(6)} = s_{14}^2 s_{123}^4 + \mathcal{P}_6, \quad b_9^{(6)} = s_{14}^4 s_{123}^2 + \mathcal{P}_6, \\
b_{10}^{(6)} &= s_{123}^2 s_{124}^2 s_{135}^2 + \mathcal{P}_6, \quad b^{(2)}_{1,S_2\times S_4} = s_{12}^2, \quad b^{(2)}_{2,S_2\times S_4} = s_{13}^2 + \mathcal{P}_{\{12|3456\}}, \\
b^{(2)}_{3,S_2\times S_4} &= s_{34}^2 + \mathcal{P}_{\{12|3456\}}, \quad b^{(2)}_{4,S_2\times S_4} = s_{12}s_{13} + \mathcal{P}_{\{12|3456\}}
\end{aligned}$$

All soft theorems are satisfied

Non-Perturbative Verifications

The instanton effective action of $\mathcal{N}=4$ SYM on the Coulomb branch, Massimo, Morales, Wen

$$S_{\rm eff}^{1-inst} = c' \, \frac{g^4}{\pi^6} \, e^{2\pi i \tau} \int \frac{d^4x \, d^8\theta \, \sqrt{\det_{4N} 2\bar{\Phi}_{Au,Bv}}}{\sqrt{\det_{2N}\left(\Phi^{AB}\bar{\Phi}_{AB} + \frac{1}{g}\bar{\mathcal{F}} + \frac{1}{\sqrt{2}g}\bar{\Lambda}_A(\Phi^{-1})^{AB}\bar{\Lambda}_B\right)_{\dot{\alpha}u,\dot{\beta}v}}} \, .$$

The $\mathcal{N}=4$ on-shell superfields can be expanded in terms of the component fields $\{\phi^{AB}, \lambda^A_\alpha, F^-_{\alpha\beta}\}$. For just the scalars,

$$\bar{\Phi}_{AB} = \bar{\phi}_{AB} \, , \quad \bar{\Lambda}_{A\dot{\alpha}} = i\,\theta^{B\alpha}\partial_{\alpha\dot{\alpha}}\bar{\phi}_{AB} \, , \quad \bar{\mathcal{F}}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\theta^{A\alpha}\theta^{B\beta}\partial_{\alpha\dot{\alpha}}\partial_{\beta\dot{\beta}}\bar{\phi}_{AB}$$

We obtain simple dilaton effective action

$$S_{\rm dilaton} = \int d^4x \, \left[(S_{\mu\nu}S^{\mu\nu})^2 - S_{\mu\nu}S^{\nu\rho}S_{\rho\sigma}S^{\sigma\mu}\right] , \qquad S_{\mu\nu} = \frac{\partial_\mu\partial_\nu\varphi}{\varphi^2} - 2\frac{\partial_\mu\varphi\partial_\nu\varphi}{\varphi^3} \, ;$$

Non-Perturbative Verifications

But horrific vertices when expanded around $\varphi \to v + \varphi$

$$v^8\Gamma^{(4)}[\varphi]=(\partial_\mu\partial_\nu\varphi\,\partial^\mu\partial^\nu\varphi)^2-\partial_\mu\partial_\nu\varphi\,\partial^\nu\partial^\rho\varphi\,\partial_\rho\partial_\sigma\varphi\,\partial^\sigma\partial^\mu\varphi \equiv (\partial\partial\varphi\cdot\partial\partial\varphi)^2-(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi) \tag{A.1}$$

$$\begin{aligned} v^9\Gamma^{(5)}[\varphi] &= -8\,\varphi(\partial\partial\varphi\cdot\partial\partial\varphi)^2+8\,\varphi(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi) \\ &\quad -8(\partial\partial\varphi\cdot\partial\partial\varphi)\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi+8\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi-2(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\;\partial\varphi\cdot\partial\varphi) \end{aligned} \tag{A.2}$$

$$\begin{aligned} v^{10}\Gamma^{(6)}[\varphi] &= 36\,\varphi^2(\partial\partial\varphi\cdot\partial\partial\varphi)^2-36\,\varphi^2(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi) \\ &\quad +72\,\varphi(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi)\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi-72\,\varphi\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi \\ &\quad +18\,\varphi(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi)+8(\partial\cdot\partial\partial\varphi\cdot\partial\varphi)^2-4\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi+3(\partial\partial\varphi\cdot\partial\partial\varphi)\,(\partial\varphi\partial\varphi)^2 \end{aligned} \tag{A.3}$$

$$\begin{aligned} v^{11}\Gamma^{(7)}[\varphi] &= -120\,\varphi^3(\partial\partial\varphi\cdot\partial\partial\varphi)^2+120\,\varphi^3(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi) \\ &\quad -360\,\varphi^2(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi)\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi+360\,\varphi^2\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi \\ &\quad -90\,\varphi^2(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi)-80\,\varphi(\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi)^2+40\,\varphi\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi \\ &\quad -45\,\varphi\,(\partial\partial\varphi\cdot\partial\partial\varphi)\,(\partial\varphi\partial\varphi)^2-10\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi\,(\partial\varphi\partial\varphi)^2 \end{aligned} \tag{A.4}$$

$$\begin{aligned} v^{12}\Gamma^{(8)}[\varphi] &= 330\,\varphi^4(\partial\partial\varphi\cdot\partial\partial\varphi)^2-330\,\varphi^4(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi) \\ &\quad +1320\,\varphi^3(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi)\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi-1320\,\varphi^3\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi \\ &\quad +330\,\varphi^3(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi)+440\,\varphi^2(\partial\cdot\partial\partial\varphi\cdot\partial\varphi)^2-220\,\varphi^2\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi \\ &\quad +\frac{495}{2}\,\varphi^2\,(\partial\partial\varphi\cdot\partial\partial\varphi)\,(\partial\varphi\partial\varphi)^2+110\,\varphi\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi\,(\partial\varphi\partial\varphi)^2+\frac{15}{4}(\partial\varphi\partial\varphi)^4 \end{aligned} \tag{A.5}$$

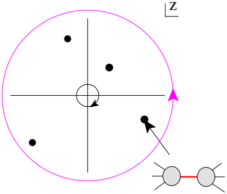
$$\begin{aligned} v^{13}\Gamma^{(9)}[\varphi] &= -792\,\varphi^5(\partial\partial\varphi\cdot\partial\partial\varphi)^2+792\,\varphi^5(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi) \\ &\quad -3960\,\varphi^4(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi)\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi+3960\,\varphi^4\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi \\ &\quad -990\,\varphi^4(\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi)-760\,\varphi^3(\partial\cdot\partial\partial\varphi\cdot\partial\varphi)^2+880\,\varphi^3\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi \\ &\quad -990\,\varphi^3\,(\partial\partial\varphi\cdot\partial\partial\varphi)\,(\partial\varphi\partial\varphi)^2-660\,\varphi^2\,\partial\varphi\cdot\partial\partial\varphi\cdot\partial\varphi\,(\partial\varphi\partial\varphi)^2-45\,\varphi\,(\partial\varphi\partial\varphi)^4. \end{aligned}$$

All soft theorems are satisfied

Constraints on effective action

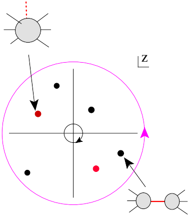
Using the fact that S-matrix are analytic functions, we start with: Britto, Cachazo, Feng, Witten

$$A_n(0) = \oint_{|z|=0} dz \frac{A_n(z)}{z} = -\oint_{|z|=z^*} dz \frac{A_n(z)}{z},$$



The constraint from soft-theorems can be utilized via augmented recursion: Cheung, Kampf, Novotny, Shen, Trnka

$$A_n(0) = \oint_{|z|=0} dz \frac{A_n(z)}{zF(z)} = -\oint_{|z|=z^*} dz \frac{A_n(z)}{zF(z)} - \oint_{|z|=z^*} dz \frac{A_n(z)}{zF(z)},$$



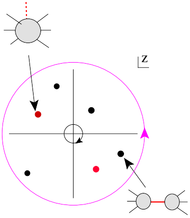
Constraints on effective action

Take

$$A(z) = A|_{p_i \to (1 - z a_i) p_i}, \quad F_n(z) = \prod_{i=1}^{n} [(1 - z a_i)]^{d_i}$$

with $\sum_i a_i p_i = 0$

$$A_n(0) = \oint_{|z|=0} dz \frac{A_n(z)}{zF(z)} = -\oint_{|z|=z^*} dz \frac{A_n(z)}{zF(z)} - \oint_{|z|=z^*} dz \frac{A_n(z)}{zF(z)},$$



The residue of $F(z)$ is determined

$$A(z) \to A_0 + A_1 q + A_2 q^2 + \cdots A_d q^{d-1}$$

where $q = (1 - z a_i) p_i$

Constraints on effective action

The residue of $F(z)$ is determined

$$A(z) \to A_0 + A_1 q + A_2 q^2 + \cdots A_d q^{d-1}$$

Since for the pure dilaton sector

$$M_n\big|_{p_n \to 0} = \left(\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(1)}\right) M_{n-1} + \mathcal{O}(p_n^2)\,,$$

we have $d = 2$.

The pure dilaton amplitude can be constructed using recursion

$$A_n(0) = \oint_{|z|=0} dz \frac{A_n(z)}{z \prod_i (1 - z a_i)^2}$$

The denominator $\sim z^{2n}$, while $A_n(z) \sim z^{2m}$ for order ∂^{2m} $\to$ we need $n > m$

Constraints on effective action

The pure dilaton sector is highly constrained:

$s^n \setminus$ # of points	4	5	6	7	8	$\cdots$
2	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
3	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
4	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
5	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
6	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
7	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$
8	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

At s^n, the EFT is determined up to coefficients for operators $\partial^{2n}\varphi^n$

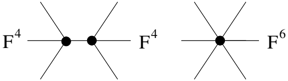
SUSY Constraints on effective action

Maximal SUSY is known to give exact results:

- s^2: F^4 operator one-loop exact $\lambda = \left(\frac{g^4 N}{8\pi^2 m^4}\right)$
- For the pure field-strengths Chen, Y-t, Wen

$$\mathcal{L}_{\text{eff}} = \sum_{p,q=1} c_0^{p,q} \frac{(F_+^2)^p (F_-^2)^q}{(M^2)^{2(p+q-1)}} + \sum_{m=1} \sum_{p,q=1} c_m^{p,q} \frac{D^{2m}(F_+^2)^p (F_-^2)^q}{(M^2)^{2(p+q-1)+m}} + \cdots$$

There are no local susy matrix elements that encode $F_-^2 F_+^{n-2}$ $\rightarrow$must have zero coefficient



One obtains an exact recursion formula

$$c_0^{1,q} = 4^{q-1}(c_0^{1,1})^q \,.$$

SUSY Constraints on effective action

Assume D=4 maximal susy

$$\mathcal{A}_4 = \delta^8(Q)\frac{[12]^2}{\langle 34\rangle^2}\sum_k P_4^{(k)}(s_{ij}),$$

$$\mathcal{A}_5 = v\,\delta^8(Q)\frac{m^{(1)}_{1,2,3}m^{(2)}_{1,2,3}+m^{(3)}_{1,2,3}m^{(4)}_{1,2,3}}{\langle 45\rangle^2}\sum_k P_5^{(k)}(s_{ij}),$$

- s^2: F^4 operator one-loop exact $\lambda = \left(\frac{g^4 N}{8\pi^2 m^4}\right)$
- s^3: $A_4^{(3)} = A_5^{(3)} = 0$, and the first non-zero would be A_6

$$\begin{aligned} A_6^{(3)} &= a_1(s_{12}^3+\mathcal{P}_6)+a_2(s_{123}^3+\mathcal{P}_6) \\ &+ \lambda^2\left((s_{12}^2+s_{13}^2+s_{23}^2)\frac{1}{s_{123}}(s_{45}^2+s_{46}^2+s_{56}^2)+\mathcal{P}_6\right) \end{aligned}$$

soft theorem fixes $a_1 = 0\,, \quad a_2 = -\lambda^2 \rightarrow A_n^{(3)}$ is two-loop exact

Up to six-derivatives, the effective action is identical to DBI in $AdS_5 \times S_5$

SUSY Constraints on effective action

- s^4: Recursion determines all $n > 4$ in terms of the four-point

$$\sum_{m\leq 8} \mathcal{L}_{\partial^m \phi^n} = \delta_{m,8}\, c_4^{(2)}(g,N)\mathcal{L}^{\ell=1}_{\partial^8\phi^n} + \sum_{m\leq 8} \mathcal{L}^{\rm DBI}_{\partial^m\phi^n}\,,$$

- s^5:

$$P_4^{(3)}(s_{ij}) = c_4^{(3)}(g,N)\times(s_{12}^3+\mathcal{P}_4)\,,\quad P_5^{(3)}(s_{ij}) = c_5^{(3)}(g,N)\times(s_{12}^3+\mathcal{P}_5)\,.$$

Soft theorem determines $c_5^{(3)}(g,N) = -c_4^{(3)}(g,N)$

$$\mathcal{L}_{\partial^{10}\phi^n} = c_4^{(3)}(g,N)\mathcal{L}^{\ell=1}_{\partial^{10}\phi^n} + \lambda\times c_4^{(2)}(g,N)\mathcal{L}^{\ell=2}_{\partial^{10}\phi^n} + \mathcal{L}^{\rm DBI}_{\partial^{10}\phi^n},$$

Maximal SUSY fixes the effective action up to 10 derivatives in terms of two unknown coefficients

Scale vs Conformal symmetry

$$M_n\big|_{p_n\to 0} = \left(\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(1)}\right) M_{n-1} + \mathcal{O}(p_n^2)\,,$$

$$\mathcal{S}_n^{(0)} = \sum_{i=1}^{n-1}\left(p_i\cdot\frac{\partial}{\partial p_i} + \frac{d-2}{2}\right) - d\,, \leftarrow \textit{Dilatation}$$

$$\mathcal{S}_n^{(1)} = p_n^{\mu}\sum_{i=1}^{n-1}\left[p_i^{\nu}\frac{\partial^2}{\partial p_i^{\nu}\partial p_i^{\mu}} - \frac{p_{i\mu}}{2}\frac{\partial^2}{\partial p_{i\nu}\partial p_i^{\nu}} + \frac{d-2}{2}\frac{\partial}{\partial p_i^{\mu}}\right] \leftarrow \textit{Conformal Boost}.$$

“To what extent does the sub-leading soft theorem, due to broken conformal boost symmetry, follow from the leading behaviour stemming from broken dilation symmetry?”

- To all order in derivative coupling, the five point matrix elements satisfying leading soft automatically satisfies subleading soft theorems.
- At s^3, we can construct a local polynomial $s_{12}^3 - 2s_{12}s_{13}s_{23} + \mathcal{P}_6$ at six-points, which vanish at leading, but no subheading soft-limit not conformal

$$L = (d\phi)^2 + (dv)^2 + g\phi^2 v^{\mu}v_{\mu}$$

Current and future directions

- Detailed study of consistent massive interactions, under unitarity+locality
- S-matrix boot-strap: for $s, t < 0$

$$\sum_{i,L} \frac{c_{i,L}^2 P_L(\cos\theta)}{s - m_i^2} = \sum_{i,L} \frac{c_{i,L}^2 P_L(\cos\theta)}{t - m_i^2}$$

  can we prove that the spectrum can be organised as $m_i^2 = \frac{i}{\alpha'_j}$

- Mysteries for double soft-limits, $[G_i, G_j] = f_{ij}{}^k G_k$:
  DBI is a truncation for conformal DBI, yet two soft theorems are completely different.
- Are soft theorems valid for UV divergences ? First one-loop 6-pt test agrees for A-V theory
- What rules out scale but not-conformal invariant theories (identified operators that signal non-unitarity in UV)